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EQUATIONS OF ELASTOVISCOPLASTIC MEDIUM WITH FINITE DEFORMATIONS

V. I. Kondaurov UDC 539.371

In this paper, we examine the nonstationary equations of the theory of flow of finitely deformed elastoviscoplastic materials. We analyze two approaches to describing the kinematics of such media. We study the restrictions imposed on the determining equations by the entropy inequality and the requirements of invariance relative to orthogonal transformations of the actual, unloaded and initial configurations. The complete system of equations is written in divergence form, which permits obtaining all allowable relations at strong discontinuities. In the adiabatic approximation, the system of equations reduces to a symmetrical form and we formulate sufficient conditions for hyperbolicity.

1. Kinematics. Let ξ be the radius vector of a particle in the medium in the initial configuration of the body and $\mathbf{x}$ the actual instantaneous configuration. We shall assume that the initial configuration is the natural configuration [1, 2] with constant temperature $\theta = \theta_0$ and density $\rho = \rho_0$ = const. We shall denote by $\overset{\circ}{e}_i, \widehat{e}_j$ the basis vectors of the starting and accompanying Lagrangian system of coordinates [1] and by e_k the basis of the spatial Cartesian coordinate system, such that

$$d\xi = d\xi^i \overset{\circ}{e}_i,\quad d\mathbf{x} = dx^k e_k = d\xi^j \widehat{e}_j. \tag{1.1}$$

We shall assume that the mapping (deformation) of the starting configuration into the actual configuration

$$\mathbf{x} = \mathbf{x}(\xi, t), \tag{1.2}$$

where t is the time, is mutually unique and continuously differentiable the required number of times. For fixed t, it follows from (1.2) that

$$d\mathbf{x} = \mathbf{F}\cdot d\xi = \left(\overset{\circ}{e}_a F^{a}_{\cdot b} \overset{\circ}{e}{}^b\right)\cdot\left(\overset{\circ}{e}_j d\xi^j\right) = \overset{\circ}{e}_a F^{a}_{\cdot j} d\xi^j, \tag{1.3}$$

where $\mathbf{F}$ is the tensor of the gradient of the total deformation. Equating (1.1) and (1.3) we see that

$$\widehat{e}_j = \overset{\circ}{e}_i F^{i}_{\cdot j}, \tag{1.4}$$

i.e., the matrix $F^{i}_{\cdot j}$ is the linear transformation of both $d\xi$ into dx and the basis $\overset{\circ}{e}_i$ into the basis $\widehat{e}_j$.

Using the definition of the velocity vector $\mathbf{v} = \partial\mathbf{x}(\xi, t)/\partial t|_\xi$ and relation (1.4), we obtain

$$d\mathbf{v} = \nabla\mathbf{v}\cdot d\mathbf{x} = d\xi^i \left.\frac{\partial \widehat{e}_i}{\partial t}\right|_{\xi^m} = d\xi^i \left.\frac{\partial F^{k}_{\cdot j}}{\partial t}\right|_{\xi^m} \overset{\circ}{e}_k = \left.\frac{\partial \mathbf{F}}{\partial t}\right|_{\xi}\cdot d\xi,$$

from where in view of the arbitrariness of $d\mathbf{x}$ follows the kinematic relation [1, 2]

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$$\dot{\mathbf{F}}\mathbf{F}^{-1} = \nabla \mathbf{v},\ \dot{F}^{i\cdot}_{\cdot j} = F^{k\cdot}_{\cdot j} \partial v^i / \partial x^k. \tag{1.5}$$

The dot here and in what follows indicates differentiation with respect to t with ξ = const. As shown in [3], relation (1.5), which is the condition for consistency of the deformation and velocity fields, can be put into divergence form:

$$\left.\frac{\partial \left(\frac{1}{\Delta} F^{i\cdot}_{\cdot j}\right)}{\partial t}\right|_{x^m} + \frac{\partial}{\partial x^k}\left\{\frac{1}{\Delta}\left(v^k F^{i\cdot}_{\cdot j} - v^i F^{k\cdot}_{\cdot j}\right)\right\} = 0,\ \Delta = \det \| F^{i\cdot}_{\cdot j} \|$$

or, taking into account the law of conservation of mass, $\rho\Delta = \rho_0$ for bodies with piecewise constant density in the initial configuration into the form:

$$\left.\frac{\partial \left(\rho F^{i\cdot}_{\cdot j}\right)}{\partial t}\right|_{x^m} + \frac{\partial}{\partial x^k}\{\rho v^k F^{i\cdot}_{\cdot j} - \rho v^i F^{k\cdot}_{\cdot j}\} = 0. \tag{1.6}$$

We shall now introduce, in addition to the initial and actual configurations of the body, an unloaded intermediate configuration [1] with temperature $\Theta = \Theta_0$. We shall denote the radius vector of a particle in the unloaded state as $\mathbf{y} = \mathbf{y}(\xi, t)$ and we shall assume, as done in most modern work on finite deformations of an elastoplastic medium (see, e.g., the review in [4]), that the differentials $d\mathbf{x}$, $d\mathbf{y}$, and $d\xi$ at t = const are related with one another by the relations

$$d\mathbf{x} = \mathbf{F}\cdot d\xi,\ d\mathbf{y} = \mathbf{P}\cdot d\xi,\ d\mathbf{x} = \mathbf{E}\cdot d\mathbf{y}. \tag{1.7}$$

It follows from (1.7)

$$\mathbf{F} = \mathbf{E}\cdot\mathbf{P}, \tag{1.8}$$

where $\mathbf{E}$ is the elastic deformation gradient that vanishes after the stresses are removed from the surface of an infinitely small volume; $\mathbf{P}$ is the gradient of the plastic, residual deformation $\det \mathbf{P} > 0$, $\det \mathbf{E} > 0$, $\mathbf{EP} \neq \mathbf{PE}$.

Now, let $\overset{*}{\mathbf{e}}_i$ be the basis of a Lagrangian system of coordinates in the space of the unloaded configuration such that

$$d\mathbf{y} = d\xi^i \overset{*}{\mathbf{e}}_i. \tag{1.9}$$

Here and in what follows, the asterisk indicates quantities relating to the unloaded state.

We shall show that in contrast to the matrix $\mathbf{F}$, which is the transformation matrix for both $d\xi$ into $d\mathbf{x}$ and $\overset{\circ}{\mathbf{e}}_i$ into $\widehat{\mathbf{e}}_j$, the elastic gradient matrix $\mathbf{E}$ is no longer the matrix describing the transformation of the basis $\widehat{\mathbf{e}}_j$ of the unloaded configuration into the basis of the actual configuration. We shall define the matrices $\mathcal{P}^{a\cdot}_{\cdot b}$ and $\mathcal{E}^{m\cdot}_{\cdot n}$ as follows:

$$\overset{*}{\mathbf{e}}_i = \overset{\circ}{\mathbf{e}}_a \mathcal{P}^{a\cdot}_{\cdot i},\ \widehat{\mathbf{e}}_j = \overset{*}{\mathbf{e}}_i \mathcal{E}^{i\cdot}_{\cdot j} = \overset{\circ}{\mathbf{e}}_a \mathcal{P}^{a\cdot}_{\cdot i} \mathcal{E}^{i\cdot}_{\cdot j} = \overset{\circ}{\mathbf{e}}_a F^{a\cdot}_{\cdot j}. \tag{1.10}$$

It follows from (1.10)

$$\mathbf{F} = \vec{\mathcal{P}}\cdot\vec{\mathcal{E}},\ F^{i\cdot}_{\cdot j} = \mathcal{P}^{i\cdot}_{\cdot a}\mathcal{E}^{a\cdot}_{\cdot j}. \tag{1.11}$$

Substituting (1.10) into (1.9), we obtain

$$d\mathbf{y} = \overset{*}{\mathbf{e}}_i d\xi^i = \overset{\circ}{\mathbf{e}}_a \mathcal{P}^{a\cdot}_{\cdot i} d\xi^i = \left(\overset{\circ}{\mathbf{e}}_a \mathcal{P}^{a\cdot}_{\cdot b} \overset{\circ}{\mathbf{e}}^b\right)\cdot\left(\overset{\circ}{\mathbf{e}}_i d\xi^i\right) = \vec{\mathcal{P}}\cdot d\xi.$$

On the other hand, $d\mathbf{y} = \mathbf{P}\cdot d\xi$ and, therefore $\vec{\mathcal{P}} = \mathbf{P}$. Comparing the compositions (1.8) and (1.11), we find, substituting $\vec{\mathcal{P}} = \mathbf{P}$ $\mathbf{E} = \vec{\mathcal{P}}\vec{\mathcal{E}}\vec{\mathcal{P}}^{-1}$, $\vec{\mathcal{E}} = \mathbf{P}^{-1}\mathbf{E}\mathbf{P}$, from where it is evident that $\mathbf{E}$ is not in general the matrix transforming the basis vectors $\overset{*}{\mathbf{e}}_i$ into $\widehat{\mathbf{e}}_j$.

A similar result is obtained if we start from the representations [15]:

$$d\mathbf{x} = \mathbf{F}\cdot d\xi,\ d\mathbf{y} = \mathbf{E}\cdot d\xi,\ d\mathbf{x} = \mathbf{P}\cdot d\mathbf{y},\ \mathbf{F} = \mathbf{P}\cdot\mathbf{E},$$

$$\widehat{\mathbf{e}}_i = \overset{\circ}{\mathbf{e}}_a F^{a\cdot}_{\cdot i},\ \overset{*}{\mathbf{e}}_j = \overset{\circ}{\mathbf{e}}_b \mathcal{E}^{b\cdot}_{\cdot j},\ \widehat{\mathbf{e}}_i = \overset{*}{\mathbf{e}}_m \mathcal{P}^{m\cdot}_{\cdot i},\ \mathbf{F} = \vec{\mathcal{E}}\cdot\vec{\mathcal{P}}.$$

In this case, $\vec{\mathcal{E}} = \mathbf{E}$, $\mathbf{P} = \vec{\mathcal{E}}\vec{\mathcal{P}}\vec{\mathcal{E}}^{-1}$, $\vec{\mathcal{P}} = \mathbf{E}^{-1}\mathbf{P}\mathbf{E}$.

As can be seen from the discussion above, in the case of elastoplastic bodies, it is necessary to distinguish the measure of elastic and plastic deformations: transformation of the differentials of the radius vectors in transforming from one configuration to another or transformation of the unit basis vectors. In the case of nonlinearly elastic materials, these transformations coincide identically.

Let us now examine how the tensors **F** and **P** transform under orthogonal transformations of the configurations. We shall denote by a tilde the values of quantities after the reference system is changed or, which is the same thing, after superposing translation and rotation on the actual configuration as a rigid body with fixed initial and unloaded configurations. From Eq. (1.7) and the relations determining the change in the reference system [2],

$$\tilde{\mathbf{x}} = \mathbf{z}(t) + \mathbf{Q}(t)(\mathbf{x} - \mathbf{z}_0), \tag{1.12}$$

where $\mathbf{z}(t) - \mathbf{z}_0$ is the translation vector and $\mathbf{z}_0$ is the radius vector of a point relative to which the rotation occurs, determined by the orthogonal tensor $\mathbf{Q}(t)$, it follows

$$\tilde{\mathbf{F}} = \mathbf{Q}\cdot\mathbf{F},\ \tilde{\mathbf{P}} = \mathbf{P}. \tag{1.13}$$

Let $\mathbf{Y} = \mathbf{Y}(t)$ be the transformation tensor, which does not change the metric of the unloaded configuration. Then,

$$\bar{\mathbf{F}} = \mathbf{F},\ \bar{\mathbf{P}} = \mathbf{Y}\cdot\mathbf{P}, \tag{1.14}$$

where the bar indicates the value of the quantity after the transformation $\mathbf{Y}(t)$ with fixed actual and initial configurations.

Finally, let $\mathbf{K}$ = const be the orthogonal tensor transforming the initial configuration. With fixed actual and unloaded configurations, we have

$$\bar{\bar{\mathbf{F}}} = \mathbf{F}\cdot\mathbf{K},\ \bar{\bar{\mathbf{P}}} = \mathbf{P}\cdot\mathbf{K}, \tag{1.15}$$

where the double bar indicates the quantity after the transformation indicated.

Let us examine the polar decompositions

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR},\ \mathbf{P} = \mathbf{HW} = \mathbf{MH}, \tag{1.16}$$

where **R** and **H** are orthogonal tensors; **U**, **V** and **W**, **M** are symmetrical positive-definite tensors. Using the theorem on the uniqueness of the polar decomposition, we obtain from (1.13)-(1.16)

$$\tilde{\mathbf{R}} = \mathbf{QR},\ \tilde{\mathbf{U}} = \mathbf{U},\ \tilde{\mathbf{V}} = \mathbf{QVQ}^T,\ \tilde{\mathbf{H}} = \mathbf{H},\ \tilde{\mathbf{W}} = \mathbf{W},\ \tilde{\mathbf{M}} = \mathbf{M}; \tag{1.17}$$

$$\bar{\mathbf{R}} = \mathbf{R},\ \bar{\mathbf{U}} = \mathbf{U},\ \bar{\mathbf{V}} = \mathbf{V},\ \bar{\mathbf{H}} = \mathbf{YH},\ \bar{\mathbf{W}} = \mathbf{W},\ \bar{\mathbf{M}} = \mathbf{YMY}^T; \tag{1.18}$$

$$\bar{\bar{\mathbf{R}}} = \mathbf{RK},\ \bar{\bar{\mathbf{U}}} = \mathbf{K}^T\mathbf{UK},\ \bar{\bar{\mathbf{V}}} = \mathbf{V},\ \bar{\bar{\mathbf{H}}} = \mathbf{HK},\ \bar{\bar{\mathbf{W}}} = \mathbf{K}^T\mathbf{WK},\ \bar{\bar{\mathbf{M}}} = \mathbf{M}. \tag{1.19}$$

2. Determining Equations. We shall examine the determining differential relations for moment-free homogeneous isotropic elastoplastic media, sensitive to the rate of deformation. Such equations are a quite good approximation for describing the basic effects observed with elastoplastic deformation of materials and automatically satisfy the principles of determinism and local action [1, 2]. Later, we shall study the restrictions imposed on the determining equations by the requirements of invariance and restrictions due to the entropy inequality.

We shall assume that the thermodynamic state of a particle is completely determined, if the following external variables are given: the tensor $\Theta > 0$, temperature $\mathbf{g} = \nabla\Theta$, and temperature gradient **F**, as well as internal variables, characterizing the change in the internal structure of the material accompanying plastic deformation: the tensor **P** and the strengthening parameter χ. We shall denote for convenience $\pi_\alpha = \{\mathbf{F}, \mathbf{P}, \Theta, \chi, \mathbf{g}\}$ and illustrate the determining equations in the simplest, functional form

$$A = A(\pi_\alpha),\ \boldsymbol{\sigma} = \Sigma(\pi_\alpha),\ \eta = \eta(\pi_\alpha),\ \mathbf{q} = \mathrm{S}(\pi_\alpha); \tag{2.1}$$

$$\Phi(\dot{\mathbf{P}}, \dot{\chi}, \pi_\alpha) = 0,\ \varphi(\dot{\mathbf{P}}, \dot{\chi}, \pi_\alpha) = 0,\ \partial(\Phi_{ij}, \varphi)/\partial(\dot{P}_{mn}, \dot{\chi}) \neq 0, \tag{2.2}$$

where A is the free energy density, σ is the Cauchy tensor, η is the entropy density, and **q** is the heat flux. The functions $A(\pi_\alpha)$, $\Sigma(\pi_\alpha)$, $\eta(\pi_\alpha)$ and $S(\pi_\alpha)$ are assumed to be sufficiently smooth. The evolutionary equations (2.2), where Φ is tensor and φ scalar functions of their arguments, are assumed to be solvable for $\dot{\mathbf{P}}$ and $\dot{\chi}$.

Let us now examine the orthogonal transformation and parallel transport of the initial configuration with fixed unloaded and actual configurations. Taking into account the fact that with this transformation the following equations are valid in addition to relations (1.19)

$$\bar{\dot{\mathbf{P}}} = \dot{\mathbf{P}}\mathbf{K},\ \bar{\dot{\chi}} = \dot{\chi}, \tag{2.3}$$

we find that a necessary and sufficient condition for isotropy and homogeneity of the elastoviscoplastic material is

$$A = A(\mathbf{V}, \mathbf{PR}^T),\ \boldsymbol{\sigma} = \boldsymbol{\Sigma}(\mathbf{V}, \mathbf{PR}^T),\ \eta = \eta(\mathbf{V}, \mathbf{PR}^T),\ \mathbf{q} = \mathbf{S}(\mathbf{V}, \mathbf{PR}^T),$$
$$\boldsymbol{\Phi}(\mathbf{V}, \mathbf{PR}^T, \dot{\mathbf{P}}\mathbf{R}^T, \dot{\chi}) = 0,\ \varphi(\mathbf{V}, \mathbf{PR}^T, \dot{\mathbf{P}}\mathbf{R}^T, \dot{\chi}) = 0, \tag{2.4}$$

where for brevity we drop the arguments $\boldsymbol{\Theta}$, χ and $\mathbf{g}$, which do not vary under the transformation being examined. The necessity of (2.4) can be easily checked if for a fixed particle and fixed time the arbitrary constant orthogonal tensor $\mathbf{K}$ is set equal to the tensor $\mathbf{R}^T$. Substituting (1.19), (2.3) into (2.4), it can be shown that (2.4) is a sufficient condition.

If we examine the condition for invariance of the determining equations (2.4) relative to orthogonal transformations of the unloaded configuration, then we obtain

$$A = A(\mathbf{V}, \mathbf{B}),\ \boldsymbol{\sigma} = \boldsymbol{\Sigma}(\mathbf{V}, \mathbf{B}),\ \eta = \eta(\mathbf{V}, \mathbf{B}),\ \mathbf{q} = \mathbf{S}(\mathbf{V}, \mathbf{B}),$$
$$\boldsymbol{\Phi}(\mathbf{V}, \mathbf{B}, \mathbf{R}\dot{\mathbf{W}}\mathbf{R}^T, \dot{\chi}) = 0,\ \varphi(\mathbf{V}, \mathbf{B}, \mathbf{R}\dot{\mathbf{W}}\mathbf{R}^T, \dot{\chi}) = 0, \tag{2.5}$$

where $\mathbf{B} = \mathbf{RWR}^T$.

The condition for invariance of equations relative to transformations of the unloaded configuration, it seems to us, completely eliminates the problem of the nonuniqueness of the decomposition $\mathbf{F} = \mathbf{E}\cdot\mathbf{P}$, discussed many times beginning with [6] and is a natural generalization of different assumptions concerning the mutual relation between the actual and unloaded configurations.

In order to prove the necessity, we shall fix the particle ξ at time $t = t_0$. Now, let the transformation $\mathbf{Y}(t)$ for $0 \leqslant \tau \leqslant t_0$ be such that

$$\mathbf{Y}(\tau) = \mathbf{R}(t_0)\mathbf{H}^T(\tau),\ \dot{\mathbf{Y}}(\tau) = \mathbf{R}(t_0)\dot{\mathbf{H}}^T(\tau). \tag{2.6}$$

At time $\tau = t_0$, taking into account (1.18) and (2.6), we have

$$\overline{\mathbf{PR}^T} = \mathbf{RWR}^T,\ \overline{\dot{\mathbf{P}}\mathbf{R}^T} = \mathbf{R}\dot{\mathbf{W}}\mathbf{R}^T,\ \overline{\mathbf{V}} = \mathbf{V},$$

from where follows the necessity of (2.5). The proof that (2.5) is sufficient is obtained if relations (1.18) are substituted into them.

Let us clarify the restrictions imposed on (2.5) by the requirement of invariance relative to orthogonal transformations of (1.12) of the actual configuration. Then, taking into account the fact that A, σ, η and $\mathbf{q}$ are object [2] quantities, we obtain, keeping in mind (1.17):

$$A(\mathbf{QVQ}^T, \mathbf{QBQ}^T, \mathbf{Qg}) = A(\mathbf{V}, \mathbf{B}, \mathbf{g}),$$
$$\boldsymbol{\Sigma}(\mathbf{QVQ}^T, \mathbf{QBQ}^T, \mathbf{Qg}) = \mathbf{Q}\boldsymbol{\Sigma}(\mathbf{V}, \mathbf{B}, \mathbf{g})\mathbf{Q}^T,$$
$$\eta(\mathbf{QVQ}^T, \mathbf{QBQ}^T, \mathbf{Qg}) = \eta(\mathbf{V}, \mathbf{B}, \mathbf{g}),\ \mathbf{S}(\mathbf{QVQ}^T, \mathbf{QBQ}^T, \mathbf{Qg}) = \mathbf{QS}(\mathbf{V}, \mathbf{B}, \mathbf{g}),$$
$$\mathbf{QR}\dot{\mathbf{W}}\mathbf{R}^T\mathbf{Q}^T = \boldsymbol{\Psi}(\mathbf{QVQ}^T, \mathbf{QBQ}^T, \mathbf{Qg}) = \mathbf{Q}\boldsymbol{\Psi}(\mathbf{V}, \mathbf{B}, \mathbf{g})\mathbf{Q}^T,$$
$$\dot{\chi} = \psi(\mathbf{QVQ}^T, \mathbf{QBQ}^T, \mathbf{Qg}) = \psi(\mathbf{V}, \mathbf{B}, \mathbf{g}), \tag{2.7}$$

where the resolved form of the laws of plastic flow and strengthening are used:

$$\mathbf{R}\dot{\mathbf{W}}\mathbf{R}^T = \boldsymbol{\Psi}(\mathbf{V}, \mathbf{B}, \mathbf{g}),\ \dot{\chi} = \psi(\mathbf{V}, \mathbf{B}, \mathbf{g}). \tag{2.8}$$

It follows from (2.7) that the functions A, Σ, η, S, Ψ and ψ are isotropic functions of their arguments. Assuming, in particular, that $Q = R^T$, we obtain

$$A = A(\mathbf{U}, \mathbf{W}, \mathbf{R}^T\mathbf{g}),\ \eta = \eta(\mathbf{U}, \mathbf{W}, \mathbf{R}^T\mathbf{g}),\ \boldsymbol{\sigma} = \mathbf{R}\boldsymbol{\Sigma}(\mathbf{U}, \mathbf{W}, \mathbf{R}^T\mathbf{g})\mathbf{R}^T,$$
$$\mathbf{q} = \mathbf{RS}(\mathbf{U}, \mathbf{W}, \mathbf{R}^T\mathbf{g}),\ \dot{\mathbf{W}} = \boldsymbol{\Psi}(\mathbf{U}, \mathbf{W}, \mathbf{R}^T\mathbf{g}),\ \dot{\chi} = \psi(\mathbf{U}, \mathbf{W}, \mathbf{R}^T\mathbf{g}). \tag{2.9}$$

The necessary and sufficient conditions of invariance formulated above do not, of course, give unique measures that should be used for deformations and rates of deformations and the specific form of the determining equations of an elastoviscoplastic medium, but they greatly narrow the class of permissible equations of state.

Let us clarify the restrictions imposed on the form of the determining equations (2.9) by the second law of thermodynamics, which we shall use in the form of the Clausius–Duheim inequality [2]: $\rho\dot{\eta} - \operatorname{div}((1/\Theta)\mathbf{q}) - (1/\Theta)\rho r \geqslant 0$.

Using the relation $A = \varepsilon - \eta\Theta$, where ε is the internal energy density, the reduced equation for energy balance $\rho\dot{\varepsilon} = \mathrm{tr}(\sigma\cdot\nabla\mathbf{v}) + \mathrm{div}\,\mathbf{q} + \rho r$ and the kinematic equation (1.5), the entropy inequality can be written in the form

$$-\mathrm{tr}\left(\frac{\partial A}{\partial \mathbf{U}}\dot{\mathbf{U}}\right)-\mathrm{tr}\left(\frac{\partial A}{\partial \mathbf{W}}\dot{\mathbf{W}}\right)-\frac{\partial A}{\partial \chi}\dot{\chi}-\frac{\partial A}{\partial \Theta}\dot{\Theta}-\frac{\partial A}{\partial \mathbf{g}}\dot{\mathbf{g}}-\eta\dot{\Theta}+\frac{1}{\rho}\mathrm{tr}\,\{(\mathbf{U}^{-1}\mathbf{R}^T\sigma\mathbf{R})\,\dot{\mathbf{U}}\}+\frac{1}{\rho\Theta}\mathbf{q}\cdot\mathbf{g}\geqslant 0. \tag{2.10}$$

For the elastoviscoplastic material examined, it is postulated that in the space $(\mathbf{U}, \mathbf{W}, \Theta, \chi, \mathbf{g})$ there exists a surface

$$f(\mathbf{U}, \mathbf{W}, \Theta, \chi, \mathbf{g}) = 0, \tag{2.11}$$

called the static condition of plasticity which separates open regions of elasticity, where $\Psi \equiv 0$, $\psi \equiv 0$, and plasticity where $\Psi \neq 0$, $\psi \neq 0$.

The values of the functions and their derivatives, determined in the elastic and plastic regions, are assumed to be continuous in the entire space, including the surface (2.11). Constructing in a standard way [2] the local continuation of the process, satisfying the law of plastic flow and strengthening, we obtain from (2.10)

$$A = A(\mathbf{U}, \mathbf{W}, \chi, \Theta),\ \partial A/\partial \mathbf{g} = 0,\ \eta = -\partial A/\partial\Theta, \tag{2.12}$$

$$\sigma = (\rho\mathbf{F}\partial A/\partial\mathbf{U})\mathbf{R}^T = \rho\mathbf{V}\partial A/\partial\mathbf{V} = \rho\mathbf{F}\partial A/\partial\mathbf{F}^T;$$

$$\delta = \mathbf{h}\cdot\mathbf{g} + \mathrm{tr}(\tau\cdot\Psi) + b\psi \geqslant 0, \text{ where} \tag{2.13}$$

$$\tau = -\partial A/\partial\mathbf{W} = -(\mathbf{R}^T\cdot\partial A/\partial\mathbf{B})\cdot\mathbf{R},\ b = -\partial A/\partial\chi,\ \mathbf{h} = (1/\rho\Theta)\mathbf{q}. \tag{2.14}$$

If we are examining an isothermal ($\mathbf{g} \equiv 0$) or adiabatic ($\mathbf{q} \equiv 0$) approximation, the Clausius–Duheim inequality transforms into the Planck inequality for the internal mechanical dissipation and (2.13) takes the form $\delta_M = \mathrm{tr}(\tau\cdot\Psi) + b\psi \geqslant 0$. If the material is located in an elastic state, then $\Psi = 0,\ \psi = \delta_M \equiv 0$ and (2.13) is the Fourier inequality for heat dissipation: $\delta_T = \mathbf{h}\cdot\mathbf{g} \geqslant 0$.

3. Complete System of Equations. We shall examine as the solution vector the set of quantities

$$\mathbf{u} = \{\Theta,\ v_i,\ F^i_{\cdot j},\ W_{ij},\ \chi\} = \{u_\alpha\},\ \alpha = 1, 2, \ldots, 20$$

In Eulerian coordinates x^i with orthonomal basis e_i, the complete system of equations can then be written in the form of differential equations

$$\frac{d\Theta}{dt} = \frac{\Theta}{\rho c_F}\frac{\partial\sigma^k_i}{\partial\Theta}\frac{\partial v^i}{\partial x^k} + \frac{1}{\rho c_F}\frac{\partial q^k}{\partial x^k} + \frac{1}{c_F}(r + r^{(p)}), \tag{3.1}$$

$$\frac{dv^i}{dt} = F^k_a\frac{\partial^2 A}{\partial F^i_a\partial F^m_n}\frac{\partial F^m_n}{\partial x^k} - F^k_a\frac{\partial\tau^{mn}}{\partial F^i_a}\frac{\partial W_{mn}}{\partial x^k} - F^k_a\frac{\partial b}{\partial F^i_a}\frac{\partial\chi}{\partial x^k} - F^k_a\frac{\partial\eta}{\partial F^i_a}\frac{\partial\Theta}{\partial x^k} + b_i,$$

$$\frac{dF^i_j}{dt} = \frac{\partial v^i}{\partial x^k}F^k_j,\ \frac{dW_{ij}}{dt} = \Psi_{ij}(u_\alpha),\ \frac{d\chi}{dt} = \psi(u_\alpha)$$

and end relations

$$A = A(U_{mn}, W_{mn}, \Theta, \chi),\ \sigma^i_j = \rho F^i_m\frac{\partial A}{\partial F^j_m}, \tag{3.2}$$

$$q^k = q^k(U_{mn}, W_{mn}, \Theta, \chi, \partial\Theta/\partial x^m),$$

$$\eta = -\frac{\partial A}{\partial\Theta},\ c_F = \Theta\frac{\partial\eta}{\partial\Theta},\ F^i_j = R^{im}U_{mj},$$

$$\tau^{mn} = -\frac{\partial A}{\partial W_{mn}},\ b = -\frac{\partial A}{\partial\chi},\ \rho = \rho_0/\det\|F^i_j\|,$$

$$r^{(p)} = \left(\tau^{mn} - \Theta\frac{\partial\tau^{mn}}{\partial\Theta}\right)\Psi_{mn} + \left(b - \Theta\frac{\partial b}{\partial\Theta}\right)\psi,$$

where b_i is the mass force, c_F is the heat capacity with constant deformations, $r^{(p)}$ is the density of heat sources due to work on irreversible deformations. It is assumed in Eqs. (3.1) that the temperature gradient $\mathbf{g} = \nabla\Theta$ has a negligible effect on the plastic properties of the material, so that $\Psi_{mn} = \Psi_{mn}(u_\alpha)$, $\psi = \psi(u_\alpha)$.

Using the temperature equation in the form of the differential law of conservation of total energy, the equation of motion in the form of the law of conservation of momentum and the kinetic equation for F^i_j in the form of the law of conservation of the consistency of deformations (1.7), and the equations of the theory of flow (2.9), the system of equations (3.1) can be written in the form of a complete system of divergence forms:

$$\partial\varphi^0_\alpha(u_\beta)/\partial t + \partial\varphi^k_\alpha(u_\beta)/\partial x^k = f_\alpha(u_\beta), \tag{3.3}$$

where $E = \varepsilon + v_i v^i/2$; $\alpha, \beta = 1, 2, \ldots, 20$; $k = 1, 2, 3$;

$$\varphi^0_\alpha = \begin{Bmatrix} \rho E \\ \rho v^i \\ \rho F^i_j \\ \rho W_{ij} \\ \rho\chi \end{Bmatrix}; \; \varphi^k_\alpha = \begin{Bmatrix} \rho E v^k - \sigma^{ik} v_i - q^k \\ \rho v^i v^k - \sigma^{ik} \\ \rho v^k F^i_j - \rho v^i F^k_j \\ \rho v^k W_{ij} \\ \rho v^k \chi \end{Bmatrix}; \; f_\alpha = \begin{Bmatrix} \rho\left(r + b^i v_i\right) \\ \rho b^i \\ 0 \\ \rho\Psi_{ij} \\ \rho\psi \end{Bmatrix}. \tag{3.4}$$

Writing the system of equations of an elastoviscoplastic medium in the form of a complete system of differential conservation laws permits determining not only the classical, but also the generalized solution [7], to obtain all allowable relations on strong discontinuity waves, and to apply the conservative method of numerical calculation to the material being studied [8]. We emphasize that for the system of equations of the theory of flow of an elastoplastic material, insensitive to the rate of deformation, the set of independent divergence forms is exhausted by the laws of conservation of energy, momentum, and consistency of deformations.

The divergence form of the complete system of equations is especially simple in the case of Lagrangian coordinates ξ^k:

$$\dot{E} - \frac{\partial}{\partial\xi^m}\left\{T^{mi}v_i + \frac{1}{\rho}F^{-1m}_k q^k\right\} = r + b^i v_i, \tag{3.5}$$

$$\dot{v}^i - \frac{\partial T^{mi}}{\partial\xi^m} = b^i,\; \dot{F}^i_j - \frac{\partial\left(v^i\delta^m_j\right)}{\partial\xi^m} = 0,\; \dot{W}_{ij} = \Psi_{ij},\; \dot{\chi} = \psi, \text{ where } T^{mi} = \frac{1}{\rho}F^{-1m}_k\sigma^{ki},\; T^{mi} \neq T^{im}.$$

In the adiabatic approximation ($q^k \equiv 0$), the system (3.5) can be symmetrized. In order to show this, we shall obtain as a result of (3.5) the equation

$$\dot{\eta} = r/\Theta + \delta,\; \delta = \tau^{mn}\Psi_{mn} + b\psi \tag{3.6}$$

for the rate of change of entropy. Expression (3.6) is an additional law of conservation, valid in regions of smooth flows. In order to derive (3.6) let us multiply the first equation (3.5) by the as yet unknown factor of $\alpha = \alpha(u_\alpha)$ and the second by β_i, the third by γ^j_i, the fourth by λ^{ji}, and the fifth by ζ and add all equations. As a result we obtain the system

$$\dot{\eta} = \alpha\dot{E} + \beta_i\dot{v}^i + \gamma^j_i\dot{F}^i_j + \lambda^{ji}\dot{W}_{ij} + \zeta\dot{\chi}, \tag{3.7}$$

$$\frac{1}{\Theta}\left(r + \tau^{mn}\Psi_{mn} + b\psi\right) = \alpha r + \alpha b^i v_i + \beta_i b^i + \lambda^{mn}\Psi_{mn} + \zeta\psi,$$

$$\alpha\frac{\partial}{\partial\xi^m}\left(T^{mi}v_i\right) + \beta_i\frac{\partial T^{mi}}{\partial\xi^m} + \gamma^j_i\frac{\partial\left(v^i\delta^m_j\right)}{\partial\xi^m} = 0,$$

from which we find

$$\alpha = \frac{1}{\Theta},\; \beta_i = -\frac{v_i}{\Theta},\; \gamma^j_i = -\frac{1}{\Theta}T^{j\cdot}_{\cdot i},\; \lambda^{ji} = \frac{1}{\Theta}\tau^{ji},\; \zeta = \frac{1}{\Theta}b. \tag{3.8}$$

Using the identity $dA = T^{m\cdot}_{\cdot i}dF^{i\cdot}_{\cdot m} - \tau^{mn}dW_{mn} - bd\chi - \eta d\Theta$, equivalent to relations (2.12) and (2.14), we can verify that expressions (3.8) satisfy the stronger, compared to (3.7), conditions:

$$d\eta = \alpha dE + \beta_i dv^i + \gamma^j_i dF^i_j + \lambda^{ji}dW_{ij} + \zeta d\chi, \tag{3.9}$$

$$\alpha d\left(T^{mi}v_i\right) + \beta_i dT^{mi} + \gamma^j_i d\left(v^i\delta^m_j\right) = 0.$$

Writing (3.9) in the form

$$d\left(\alpha E+\beta_i v^i+\gamma_i^j F_j^i+\lambda^{ji}W_{ij}+\zeta\chi-\eta\right)=E d\alpha+v^i d\beta_i+W_{ij}d\lambda^{ji}+\chi d\zeta,$$
$$d\left(\alpha T^{mi}v_i+\beta_i T^{mi}+\gamma_i^m v^i\right)=T^{mi}v_i d\alpha+T^{mi}d\beta_i+v^i d\gamma_i^m$$

and denoting

$$L^0=-\alpha E-\beta_i v^i-\gamma_i^j F_j^i-\lambda^{ji}W_{ij}-\zeta\chi+\eta=\frac{1}{\Theta}\left(\frac{1}{2}v_i v^i+\frac{1}{\rho}\sigma_k^k-\tau^{ij}W_{ij}-b\chi-A\right),\ L^m=-(1/\Theta)T_{\cdot i}^{m\cdot}v_i,$$

we obtain

$$E=-\partial L^0/\partial\alpha,\ v^i=-\partial L^0/\partial\beta_i,\ F_j^i=-\partial L^0/\partial\gamma_i^j,\ W_{ij}=-\partial L^0/\partial\lambda^{ji},\ \chi=-\partial L^0/\partial\zeta,$$
$$T^{mi}v_i=\partial L^m/\partial\alpha,\ T^{mi}=\partial L^m/\partial\beta_i,\ v^i\delta_j^m=\partial L^m/\partial\gamma_i^j. \tag{3.10}$$

Relations (3.10) permit writing the system (3.5) in symmetrical form

$$\frac{\partial L_{q_\alpha}^0}{\partial t}+\frac{\partial L_{q_\alpha}^m}{\partial\xi^m}=L_{q_\alpha q_\beta}^0\frac{\partial q_\beta}{\partial t}+L_{q_\alpha q_\beta}^m\frac{\partial q_\beta}{\partial\xi^m}=-\frac{1}{\rho}f_\alpha, \tag{3.11}$$

where $q_\alpha=\{\alpha,\beta_i,\gamma_i^j,\lambda^{ji},\zeta\}$; $L_{q_\alpha}^{0,m}=\partial L^{0,m}/\partial q_\alpha$; and $L_{q_\alpha q_\beta}^{0,m}=\partial^2 L^{0,m}/\partial q_\alpha\partial q_\beta$, under the condition that $\det\|\partial q_\alpha/\partial u_\beta\|\neq 0$. If the matrix $\overset{\circ}{L}_{q_\alpha q_\beta}$ is positive-definite, then the system (3.11) will be hyperbolic [9]. The condition that the matrix $\overset{\circ}{L}_{q_\alpha q_\beta}$ be positive-definite is equivalent to the condition of convexity of the function of internal energy ε with respect to its arguments: $(\partial^2\varepsilon(p_\alpha)/\partial p_\alpha\partial p_\beta)\lambda^\alpha\lambda^\beta>0,\ \forall\lambda^\alpha\neq 0,\quad \alpha,\ \beta=1,\ 2,\ \ldots,\ 20$, where $p_\alpha=\{F_j^i,\ W_{ij},\ \chi,\ \eta\}$. The proof of the equivalence is equivalent to that in [3] for the case of nonlinear elasticity.

We note that a sufficient condition for hyperbolicity is much stronger than the necessary condition:

$$\left(F_a^k n_k\right)\frac{\partial^2\varepsilon\left(F_n^m,\ W_{mn},\ \chi,\ \eta\right)}{\partial F_a^i\partial F_b^j}\left(F_b^s n_s\right)\lambda^i\lambda^j>0,\ \forall\lambda^i\neq 0,$$

which, as can be shown, occurs for the system of equations being studied.

Let us examine, in conclusion, the relations at strong discontinuities in the elasto-viscoplastic medium. Let $D=-\frac{\partial\varphi}{\partial t}\Big/\left(\frac{\partial\varphi}{\partial x^m}\frac{\partial\varphi}{\partial x^m}\right)^{1/2}$ be the velocity of the surface of discontinuity $\varphi(x^m,\ t)=0$, and let $n_i=\frac{\partial\varphi}{\partial x^i}\Big/\left(\frac{\partial\varphi}{\partial x^m}\frac{\partial\varphi}{\partial x^m}\right)^{1/2}$ be the components of the unit spatial normal oriented along the direction of motion. Then, at a strong discontinuity, for the system of equations (3.3), relations [7] $-D\left[\varphi_\alpha^0\right]+n_i\left[\varphi_\alpha^i\right]=0$ are valid, where $[a]=a^+-a^-$ is the jump in the quantity a. The indices (±) indicate the state of a particle after and in front of the front. Denoting by $G=D-n_i v^i$ the velocity of propagation of the surface $\varphi=0$ relative to the particle of the medium and using (3.4), we obtain

$$[\rho GE]+[\sigma^{ik}v_i]n_k+[q^k]n_k=0,\ [\rho Gv^i]+[\sigma^{ik}]n_k=0,$$
$$\left[\rho GF_j^i\right]+\left[\rho v^i F_j^k\right]n_k=0,\ [\rho GW_{ij}]=0,\ [\rho G\chi]=0. \tag{3.12}$$

Contracting the equation for the jump in F_j^i with the normal n_i, we find that for $D\neq 0$

$$\left[\rho F_j^k\right]n_k=0. \tag{3.13}$$

The meaning of relations (3.13) becomes clear, if the relation between the components of the normal to the surface $\varphi=0$ in the initial and actual configurations is kept in mind:

$$\overset{\circ}{n}_j=\frac{dS}{d\overset{\circ}{S}}\frac{\rho}{\overset{\circ}{\rho}}F_{\cdot j}^{k\cdot}n_k,$$

where dS, $d\overset{\circ}{S}$ are the elementary surface areas on the surface of discontinuity in the configurations examined. It follows from here that (3.13) expresses the continuity of the normal $\overset{\circ}{\mathbf{n}}$ in the initial configuration.

Using (3.13), it can be shown [3] that the condition of continuity of the mass flow is satisfied:

$$[\rho G]=0. \tag{3.14}$$

Equations (3.13) and (3.14) make it possible to write an equation for the jump in F^i_j in the form

$$[F^i_j] = h^i \rho F^k_j n_k, \; h^i = -\frac{1}{\rho G}[v^i],$$

after which we shall represent the remaining equations (3.12) in the form

$$\rho G\left\{[\varepsilon] - \frac{1}{2}\left(\sigma^{k+}_i + \sigma^{k-}_i\right) h^i n_k\right\} + [q^k]\, n_k = 0, \tag{3.15}$$
$$[\sigma^{ik}]n_k - (\rho G)^2 h^i = 0, \; \rho G[W_{ij}] = 0, \; \rho G[\chi] = 0.$$

On a contact discontinuity (G = 0) $[v^i] = [\sigma^{ik}]n_k = [q^k]n_k = 0$, while the quantities $[W_{ij}]$, $[\chi]$, $[\varepsilon]$ and $[F^i_j]$ are arbitrary. In the case of a shock wave ($G \neq 0$), it follows from (3.15) that the symmetrical part W_{ij} of the plastic gradient and the strengthening parameter χ are continuous, while the remaining quantities are discontinuous.

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ALLOWANCE FOR DIFFERENCES IN STRAIN RESISTANCE IN THE CREEP OF ISOTROPIC AND ANISOTROPIC MATERIALS

A. A. Zolochevskii

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The behavior of many light alloys and also polymer, composite, and other materials under creep conditions is characterized by differences in strain resistance. This property usually manifests itself in conventional tensile, compressive, and torsional tests.

The classic creep theory for isotropic media, based on the Mises number, does not account for differences in strain resistance. It does not distinguish between tensile and compressive strain resistance characteristics and admits the possibility of analytical description of shearing strain on the basis of the characteristics determined in tensile tests in spite of the fact that it differs fundamentally from linear strain. Equal tensile, compressive, or shearing strength characteristics are ascribed to materials whose creep is satisfactorily described within the framework of the above model. In the opposite case, differences in resistance to these two or three types of strain are contemplated. Generally, the tensile, compressive, and shearing strain resistance characteristics should obviously be considered as three mutually independent characteristics of materials.

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